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An extension theorem for t-designs ☆

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Abstract

In 1994, van Trung (Discrete Math. 128 (1994) 337–348) [9] proved that if, for some positive integers d and h, there exists an $S_\lambda(t,k,v)$ such that

$$\frac{\lambda_0^2}{v-k+1}\sum_{\substack{0\leqslant l<d\\ k-h\leqslant l\leqslant k}}\binom{k}{l}\binom{v-k+1}{k-l}((v-k)^2+(v-l))<\binom{v}{k},$$

then there exists an $S_{\lambda(v-t+1)}(t,k,v+1)$ having $v+1$ pairwise disjoint subdesigns $S_\lambda(t,k,v)$. Moreover, if B_i and B_j are any two blocks belonging to two distinct such subdesigns, then $d\leqslant|B_i\cap B_j|<k-h$. In 1999, Baudelet and Sebille (J. Combin. Des. 7 (1999) 107–112) proved that if, for some positive integers, there exists an $S_\lambda(t,k,v)$ such that

$$\lambda_0\sum_{i=1}^{m}\binom{v}{i}\binom{s}{i}\lambda_{0,n}<\binom{v}{k},$$

where $m=\min\{s,v-k\}$ and $n=\min\{i,t\}$, then there exists an

$$S_{\lambda\binom{v-t+s}{s}}(t,k,v+s)$$

having $\binom{v+s}{s}$ pairwise disjoint subdesigns $S_\lambda(t,k,v)$. The purpose of this paper is to generalize these two constructions in order to produce a new recursive construction of t-designs and a new extension theorem of t-designs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A t-(v,k,λ) *design* (or simply t-*design*), denoted by $S_\lambda(t,k,v)$, is a pair $(X,\mathscr{B})$ where X is a set of v elements called *points* and $\mathscr{B}$ is a collection of k-subsets of X (called

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blocks) such that every t-subset of X is contained in λ blocks of $\mathscr{B}$. A t-design with $\lambda=1$ is called a *Steiner system* and denoted simply by $S(t,k,v)$. A t-design is *trivial* if every k-subset of X is a block of $\mathscr{B}$. All the designs considered in this paper will be non-trivial and *simple*, that is without repeated blocks.

In a t-design $(X,\mathscr{B})$, let I be a set of i points, J a set of j points $(I\cap J=\emptyset)$ and $\lambda_{i,j}$ the number of blocks containing I and disjoint from J; it is well-known that

$$\lambda_{i,j}=\lambda\binom{v-i-j}{k-i}\Big/\binom{v-t}{k-t}\quad \text{for } 0\leqslant i+j\leqslant t.$$

$\lambda_{i,0}$ is denoted simply by λ_i (note that $\lambda_t=\lambda$ and that λ_0 is the total number of blocks of the design). It follows that any $S_\lambda(t,k,v)$ is also an $S_{\lambda_i}(i,k,v)$ for $0\leqslant i\leqslant t$. An $S_\lambda(t,k,v)$ in which $i\leqslant|B\cap B'|<j$ for all $B,B'\in\mathscr{B}$ will be denoted by $S_\lambda(t,k,v)\mathrm{int}(i,j)$; for example, any simple t-design is an $S_\lambda(t,k,v)\mathrm{int}(0,k)$. If Y is a non-empty subset of X of size $m\leqslant t$, we define

$$\mathscr{B}^Y=\{B\backslash Y\mid B\in\mathscr{B} \text{ and } Y\subset B\}$$

and

$$\mathscr{B}_Y=\{B\mid B\in\mathscr{B} \text{ and } B\cap Y=\emptyset\}.$$

It is well-known that $(X\backslash Y,\mathscr{B}^Y)$ is an $S_\lambda(t-m,k-m,v-m)$ called an mth *derived design* of the $S_\lambda(t,k,v)$, and that $(X\backslash Y,\mathscr{B}_Y)$ is an

$$S_{\lambda\binom{v-t}{k-t+m}/\binom{v-t}{k-t}}(t-m,k,v-m)$$

called an mth *residual design* of the $S_\lambda(t,k,v)$.

A design is *resolvable* if $\mathscr{B}$ is the union of pairwise disjoint $S(1,k,v)$'s called *parallel classes*. Such a partition of $\mathscr{B}$ is called a *resolution*. Two resolutions $\{P_i\}_{1\leqslant i\leqslant n}$ and $\{Q_j\}_{1\leqslant j\leqslant n}$ of the same design are said to be *mutually orthogonal* iff P_i and Q_j have at most one block in common for every $i,j\in\{1,\dots,n\}$.

An $S_{\lambda'}(t',k,v')$ $(X',\mathscr{B}')$ is a *subdesign* of an $S_\lambda(t,k,v)(X,\mathscr{B})$ $(v'\leqslant v)$ iff $X'\subset X$ and $\mathscr{B}\subset\mathscr{B}'$. In a resolvable design, the parallel classes are subdesigns. Two subdesigns $(X',\mathscr{B}')$ and $(X'',\mathscr{B}'')$ are said to be *disjoint* iff $\mathscr{B}'\cap\mathscr{B}''=\emptyset$.

In 1987, Teirlinck [7] has proved the existence of a t-design for every $t\geqslant1$, but the construction of t-designs when $t\geqslant4$ is still a difficult problem. In order to construct such designs, there are two well-known methods, the *recursive constructions* which produce a t-design from some other t-designs and the *extension theorems* which produce a t'-design from some t-designs $(t<t')$. The following constructions are well-known:

Theorem 1 (van Trung [9]). *If, for some positive integers d and h, there exists an $S_\lambda(t,k,v)$ such that*

$$\frac{\lambda_0^2}{v-k+1}\sum_{\substack{0\leqslant l<d\\ k-h\leqslant l\leqslant k}}\binom{k}{l}\binom{v-k+1}{k-l}\left((v-k)^2+(v-l)\right)<\binom{v}{k},$$

then there exists an $S_{\lambda(v-t+1)}(t,k,v+1)$ *having* $v+1$ *pairwise disjoint subdesigns* $S_\lambda(t,k,v)$. *Moreover, if* B_i *and* B_j *are any two blocks belonging to two distinct such subdesigns, then* $d \leqslant |B_i \cap B_j| < k-h$.

Theorem 2 (Baudelet and Sebille [2]). *If, for some integer* s, *there exists an* $S_\lambda(t,k,v)$ *such that*

$$\lambda_0 \sum_{i=1}^{m} \binom{v}{i}\binom{s}{i}\lambda_{0,n} < \binom{v}{k},$$

where $m=\min\{s, v-k\}$ *and* $n=\min\{i,t\}$, *then there exists an*

$$S_{\lambda\binom{v-t+s}{s}}(t,k,v+s)$$

having $\binom{v+s}{s}$ *pairwise disjoint subdesigns* $S_\lambda(t,k,v)$.

Theorem 3 (Alltop [1]). *Any resolvable* $S_\lambda(t,k,2k)$ *with* t *even is also a resolvable* $S_{\lambda(k-t)/(v-t)}(t+1,k,2k)$.

In the rest of this paper, we will always assume that

(i) $X=\{x_1,x_2,\ldots,x_{v+s}\}$,
(ii) m, n and s are integers such that $0<m\leqslant\min(n,s)$,
(iii) $X_{i_1i_2\ldots i_s}=X\backslash\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}$ where $1\leqslant i_1<i_2<\cdots<i_s\leqslant v+s$,
(iv) $(X_{i_1i_2\ldots i_s},\mathscr{B}_{i_1i_2\ldots i_s})$ is an $S_\lambda(t,k,v)$,
(v) $\mathscr{B}=\bigcup_{i_1i_2\ldots i_s}\mathscr{B}_{i_1i_2\ldots i_s}$,
(vi) $\mathscr{B}_{i_1i_2\ldots i_s}\cap\mathscr{B}_{j_1j_2\ldots j_s}\neq\emptyset$ iff $\{i_1,i_2,\ldots,i_s\}=\{j_1,j_2,\ldots,j_s\}$.

The purpose of this paper is to generalize Theorems 1 and 2, and to prove the following:

Theorem 4. *If* $(X,\mathscr{B})$ *is an* $S_{\lambda\binom{v-t+s}{s}}(t,k,v+s)$ *such that*

(i) *t is even*,
(ii) *for every* $\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}\subset X$, $(X_{i_1i_2\ldots i_s},\mathscr{B}_{i_1i_2\ldots i_s})$ *is a resolvable* $S_\lambda(t,k,v)$ *having exactly* q *mutually orthogonal resolutions*,
(iii) *if* $B\in\mathscr{B}_{i_1i_2\ldots i_s}$, $B'\in\mathscr{B}_{j_1j_2\ldots j_s}$ *and* $\{i_1,i_2,\ldots,i_s\}\neq\{j_1,j_2,\ldots,j_s\}$, *then* $d\leqslant|B\cap B'|<k-n$,
(iv) $m|s$, $s/m=v/k$ *and* $2(k+m)=v+s$,
(v) *the unique trivial* $S(m,m,s)$ *has exactly* q' *mutually orthogonal resolutions*,

then there exists a resolvable $S_{\lambda'}(t+1,k+m,v+s)$ *having at least* qq' *mutually orthogonal resolutions, with*

$$\lambda'=\lambda(k+m-t)\binom{v-t+s}{s}\binom{s}{m}\binom{k+m}{t}\Big/(v+s-t)\binom{k}{t}.$$

Moreover, this design has exactly qq' *mutually orthogonal resolutions if* $d>0$.

Although the conditions imposed on $(X,\mathscr{B})$ look very strong, we will see that such designs can actually be constructed using theorems of van Trung [8,9], Magliveras and Plambeck [6], Baudelet and Sebille [2], as well as Theorem 5 of this paper.

2. Some recursive constructions

We will use the following:

Lemma 1 (van Trung [9]). *Let $(P,\mathscr{D})$ be an $S_\lambda(t,k,v)$, let D be a d-subset of P and let $B \in \mathscr{D}$. Then the number of permutations $\sigma \in \mathrm{Sym}(P)$ such that $|D^\sigma \cap B| = l$ where $0 \leqslant l \leqslant \min\{d,k\}$ is equal to*

$$\binom{d}{l}\binom{k}{l}\binom{v-k}{d-l} l!(d-l)!(v-d)!$$

Theorem 5. *If, for some positive integers d and h, there exists an $S_\lambda(t,k,v)$ such that*

$$\lambda_0 \sum_{n=0}^{s-1}\binom{s}{n}\binom{v}{n}\sum_{n'=0}^{s-1}\binom{n}{n'}\lambda_{n',t-n'}\sum_{\substack{0\leqslant l<d\\ k-h\leqslant l\leqslant k}}\binom{v-k+n'}{k-l}\binom{k-n'}{l}$$
$$<\binom{v}{k},$$

then there exists an

$$S_{\lambda\binom{v-t+s}{s}}(t,k,v+s) \text{ having } \binom{v+s}{s}$$

pairwise disjoint subdesigns $S_\lambda(t,k,v)$. Moreover, if B_i and B_j are any two blocks belonging to two distinct such subdesigns, then $d \leqslant |B_i \cap B_j| < k-h$. Note that if $n' > t$, then $\lambda_{n',t-n'} = \lambda_t$.

Proof. We will first examine under which conditions we can construct $\binom{v+s}{s}$ pairwise disjoint $S_\lambda(t,k,v)$ designs $(X_{i_1i_2\ldots i_s},\mathscr{B}_{i_1i_2\ldots i_s})$ such that if $B \in \mathscr{B}_{i_1i_2\ldots i_s}$ and $B' \in \mathscr{B}_{j_1j_2\ldots j_s}$ with $\{i_1,i_2,\ldots,i_s\} \neq \{j_1,j_2,\ldots,j_s\}$, then $d \leqslant |B\cap B'| < k-h$. For a fixed $\mathscr{B}_{i_1i_2\ldots i_s}$, let us try to find an upper bound for the number of permutations g of the points of $X_{j_1j_2\ldots j_s}$ for which there is an l-subset L of $B \in \mathscr{B}_{j_1j_2\ldots j_s}(l \in [0,d[\cup[k-h,k])$ such that $g(L) \subset B'$ for some $B' \in \mathscr{B}_{i_1i_2\ldots i_s}$.

Suppose that $\{j_1,j_2,\ldots,j_s\}$ has exactly $s-n$ elements in $\{i_1,i_2,\ldots,i_s\}$ and n elements outside. There are $\binom{s}{n}\binom{v}{n}$ such sets and at most $\binom{n}{n'}\lambda_{n',t-n'}$ blocks of $(X_{i_1i_2\ldots i_s},\mathscr{B}_{i_1i_2\ldots i_s})$ intersecting $\{j_1,j_2,\ldots,j_s\}\backslash\{i_1,i_2,\ldots,i_s\}$ in n' points. Thus, using Lemma 1, there are at most

$$\lambda_0 \sum_{n'=0}^{n} \binom{n}{n'} \lambda_{n',t-n'} \sum_{\substack{0 \leqslant l < d \\ k-h \leqslant l \leqslant k}} \binom{k-n'}{l} \binom{k}{l} \binom{v-k}{k-n'-l}$$
$$l!(k-n'-l)!(v-k+n')!$$

permutations of $X_{j_1 j_2 \dots j_s}$ which map at least one l-subset of a block of $\mathscr{B}_{j_1 j_2 \dots j_s}$ onto an l-subset of a block of $\mathscr{B}_{i_1 i_2 \dots i_s}$. Therefore, in order to be able to construct $\binom{v+s}{s}$ pairwise disjoint such t-designs $(X_{i_1 i_2 \dots i_s}, \mathscr{B}_{i_1 i_2 \dots i_s})$, it suffices that

$$\lambda_0 \sum_{n=0}^{s-1} \binom{s}{n} \binom{v}{n} \sum_{n'=0}^{n} \binom{n}{n'} \lambda_{n',t-n'} f(d,h) < v!,$$

where

$$f(d,h) = \sum_{\substack{0 \leqslant l < d \\ k-h \leqslant l \leqslant k}} \binom{k-n'}{l} \binom{k}{l} \binom{v-k}{k-n'-l}$$
$$l!(k-n'-l)!(v-k+n')!,$$

that is

$$\lambda_0 \sum_{n=0}^{s-1} \binom{s}{n} \binom{v}{n} \sum_{n'=0}^{s-1} \lambda_{n',t-n'} \sum_{\substack{0 \leqslant l < d \\ k-h \leqslant l \leqslant k}} \binom{v-k+n'}{k-l} \binom{k-n'}{l} < \binom{v}{k}.$$

Since any t-subset of X is contained in $\binom{v-t+s}{s}$ sets $X_{i_1 i_2 \dots i_s}$, it follows that $(X, \mathscr{B})$ is an

$$S_{\lambda \binom{v-t+s}{s}}(t,k,v+s). \qquad \square$$

It is then easy to prove the following:

Corollary 1. *If there exists an $S_\lambda(t,k,v)\mathrm{int}(d,k-h)$ such that*

$$\lambda_0 \sum_{n=0}^{s-1} \binom{s}{n} \binom{v}{n} \sum_{n'=0}^{s-1} \binom{n}{n'} \lambda_{n',t-n'} \sum_{\substack{0 \leqslant l < d \\ k-h \leqslant l \leqslant k}} \binom{v-k+n'}{k-l} \binom{k-n'}{l}$$
$$< \binom{v}{k},$$

then there exists an $S_{\lambda \binom{v-t+s}{s}}(t,k,v+s)\mathrm{int}(d,k)$ having $\binom{v+s}{s}$ pairwise disjoint subdesigns $S_\lambda(t,k,v)$. Note that if $n' > t$, then $\lambda_{n',t-n'} = \lambda_t$.

The designs constructed by Theorem 5 and Corollary 1 will now be used to produce other designs:

Theorem 6. *If* $(X,\mathscr{B})$ *is an*

$$S_{\lambda\binom{v-t+s}{s}}(t,k,v+s)$$

such that

(i) $(X_{i_1i_2\ldots i_s},\mathscr{B}_{i_1i_2\ldots i_s})$ *is an* $S_\lambda(t,k,v)\mathrm{int}(d,k)$ *for all* $\{i_1,i_2,\ldots,i_s\}$,
(ii) *if* $B\in\mathscr{B}_{i_1i_2\ldots i_s}$, $B'\in\mathscr{B}_{j_1j_2\ldots j_s}$ *and* $\{i_1,i_2,\ldots,i_s\}\neq\{j_1,j_2,\ldots,j_s\}$, *then* $d\leqslant|B\cap B'|<k-n$,

then there exists an $S_{\lambda'}(t,k+m,v+s)\mathrm{int}(d,k)$ *where* $\lambda'=\lambda\binom{v-t+s}{s}\binom{s}{m}\binom{k+m}{t}/\binom{k}{t}$.

Proof. We will construct a new set of blocks $\mathscr{B}'$ on the set X. The blocks of $\mathscr{B}'$ are all the unions of a block of $\mathscr{B}_{i_1i_2\ldots i_s}$ with a block of the trivial $S(m,m,s)$ constructed on $\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}$, for all choices of $\{i_1,i_2,\ldots,i_s\}$. In other words,

$$\mathscr{B}'=\{B\cup Y\mid B\in\mathscr{B}_{i_1i_2\ldots i_s},Y\subset\{x_{i_1},x_{i_2},\ldots,x_{i_s}\},\ |Y|=m\quad\forall\{i_1,i_2,\ldots,i_s\}\}.$$

There are $\binom{v+s-t}{v-i}\binom{t}{i}$ sets $X_{i_1i_2\ldots i_s}$ which intersect a given t-subset of X in i points. Therefore $(X,\mathscr{B}')$ is a t-design, because any t-subset of X is contained in

$$\sum_{i=0}^{t}\binom{v+s-t}{v-i}\binom{t}{i}\lambda_i\binom{s-t+i}{m-t+i}$$

blocks of $\mathscr{B}'$. We have still to check that this design is simple. If C and C' are two blocks of $\mathscr{B}'$, $C=B\cup\{y_1,\ldots,y_m\}$ where $B\in\mathscr{B}_{i_1\ldots i_s}$ and $\{y_1,\ldots,y_m\}\subset\{x_{i_1},\ldots,x_{i_s}\}$, and $C'=B'\cup\{y'_1,\ldots,y'_m\}$ where $B'\in\mathscr{B}_{j_1\ldots j_s}$ and $\{y'_1,\ldots,y'_m\}\subset\{x_{j_1},\ldots,x_{j_s}\}$. If $\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}=\{x_{j_1},x_{j_2},\ldots,x_{j_s}\}$, then clearly $C\neq C'$ and $|C\cap C'|\geqslant d$ because $|B\cap B'|\geqslant d$. If $\{x_{i_1},x_{i_2},\ldots,x_{i_s}\}\neq\{x_{j_1},x_{j_2},\ldots,x_{j_s}\}$, then

$$|B'\cap C|=|B'\cap B|+|B'\cap\{y_1,y_2,\ldots,y_m\}|<(k-n)+m\leqslant k$$

and

$$|B'\cap C|=|B'\cap B|+|B'\cap\{y_1,y_2,\ldots,y_m\}|\geqslant d.$$

Thus the design is simple and any two blocks intersect in at least d points. Moreover,

$$|\mathscr{B}'|=\binom{s}{m}|\mathscr{B}|,$$

and so

$$\frac{\lambda'\binom{v+s}{t}}{\binom{k+m}{t}}=\binom{s}{m}\lambda\frac{\binom{v+s}{t}}{\binom{k}{t}}\binom{v-t+s}{s},$$

that is

$$\lambda'=\lambda\frac{\binom{s}{m}\binom{v-t+s}{s}\binom{k+m}{t}}{\binom{k}{t}}.\qquad\square$$

3. Resolvability and extension theorem

In this section, we will prove that the resolvability of the $S_\lambda(t,k,v)$ $(X_{i_1 i_2 \dots i_s}, \mathscr{B}_{i_1 i_2 \dots i_s})$ can force the resolvability of the $S_{\lambda'}(t,k+m,v+s)$ $(X,\mathscr{B})$.

Theorem 7. *If* $(X,\mathscr{B})$ *is an*

$$S_{\lambda\binom{v-t+s}{s}}(t,k,v+s)$$

such that

(i) *for all choices of* $\{i_1,i_2,\dots,i_s\}$, $(X_{i_1 i_2 \dots i_s}, \mathscr{B}_{i_1 i_2 \dots i_s})$ *is a resolvable* $S_\lambda(t,k,v)$ *having exactly* q *mutually orthogonal resolutions*,
(ii) *if* $B \in \mathscr{B}_{i_1 i_2 \dots i_s}$, $B' \in \mathscr{B}_{j_1 j_2 \dots j_s}$ *and* $\{i_1,i_2,\dots,i_s\} \neq \{j_1,j_2,\dots,j_s\}$, *then* $d \leqslant |B \cap B'| < k-n$,
(iii) $m|s$ *and* $s/m = v/k = u$,
(iv) *the unique trivial* $S(m,m,s)$ *has exactly* q' *mutually orthogonal resolutions*,

then there exists a resolvable $S_{\lambda'}(t,k+m,v+s)$ *having at least* qq' *mutually orthogonal resolutions*, *with*

$$\lambda' = \frac{\lambda\binom{v-t+s}{s}\binom{s}{m}\binom{k+m}{t}}{\binom{k}{t}}.$$

Moreover, *this design has exactly* qq' *mutually orthogonal resolutions if* $d > 0$.

Proof. The existence of the $S_{\lambda'}(t,k+m,v+s)$ follows from Theorem 6. Whenever we choose a parallel class $\{B_1,B_2,\dots,B_u\}$ in one of the designs $(X_{i_1 i_2 \dots i_s}, \mathscr{B}_{i_1 i_2 \dots i_s})$ and a parallel class $\{B'_1,B'_2,\dots,B'_u\}$ in the $S(m,m,s)$ constructed on $\{x_{i_1},x_{i_2},\dots,x_{i_s}\}$, we can construct a parallel class $\{B_1 \cup B'_1, B_2 \cup B'_2, \dots, B_u \cup B'_u\}$ in the $S_{\lambda'}(t,k+m,v+s)$. Using this argument, we can clearly construct qq' orthogonal resolutions. In order to construct another resolution, we must use a block of a design $(X_{i_1 i_2 \dots i_s}, \mathscr{B}_{i_1 i_2 \dots i_s})$ and a block of a design $(X_{j_1 j_2 \dots j_s}, \mathscr{B}_{j_1 j_2 \dots j_s})$ with $\{i_1,i_2,\dots,i_s\} \neq \{j_1,j_2,\dots,j_s\}$, which is possible only if $d=0$. □

We can now prove our extension theorem.

Proof of Theorem 4. If t is even and if $2(k+m)=v+s$, then using Theorem 3, the $S_{\lambda'}(t,k+m,v+s)$ of Theorem 7 is a resolvable $S_{\lambda''}(t+1,k+m,v+s)$ with $\lambda'' = \lambda(k+m-t)\binom{v-t+s}{s}\binom{s}{m}\binom{k+m}{t}/(v+s-t)\binom{k}{t}$. □

4. Applications

It is well-known (see for example [3]) that the existence of an Hadamard matrix of order $4a$, implies the existence of a resolvable $S_{2a-1}(2,2a,4a)$. Then the inequality of

Theorem 5 with $h = s = 2$ becomes

$$(8a-2)\sum_{n=0}^{1}\binom{2}{n}\binom{4a}{n}\sum_{n'=0}^{1}\binom{n}{n'}\lambda_{n',2-n'}\sum_{l=2a-2}^{2a}\binom{2a+n'}{2a-l}\binom{2a-n'}{l}$$
$$<\binom{4a}{2a},$$

that is

$$512a^7+192a^6+464a^5-88a^4+268a^3-182a^2+20a+2<\binom{4a}{2a}.$$

This inequality is clearly satisfied when a is sufficiently large. Therefore, if there exists an Hadamard matrix of order $4a$ with a sufficiently large, then there exists an $S_{2a(2a-1)(4a-1)}(2, 2a, 4a+2)$ satisfying the hypotheses of Theorem 4, from which we deduce the existence of a resolvable $S_{(4a-1)(4a^2-1)}(3, 2a+1, 4a+2)$.

Note that an infinite number of Hadamard matrices is known and that all Hadamard matrices of order $4a$ with $a < 107$ exist. For a survey on this subject, see [4].

5. Uncited reference

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